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Edge coloring of graphs with small maximum degrees[☆]

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ABSTRACT

By applying a discharging method, we give new lower bounds for the sizes of edge chromatic critical graphs for small maximum degrees. Furthermore, it is also proved that if G is a graph embeddable in a surface S with characteristic $c_S = -4$ or -5 or -6, then G is class one if its maximum degree $\Delta \geq 10$ or 11 or 12 respectively.

1. Introduction

Let V and E be the vertex set and edge set of a simple graph G, respectively. A k-vertex (or $(\leq k)$-vertex, $(\geq k)$-vertex) is a vertex of degree k (or $\leq k$, $\geq k$, respectively). $d_k(x)$ (or $d_{\geq k}(x)$, $d_{\leq k}(x)$) is the number of k-vertices (or $(\geq k)$-vertices, $(\leq k)$-vertices, respectively) adjacent to x. Denote by q the *average degree* of G with maximum degree Δ. For a vertex $x \in V(G)$, let $N(x)$ be the set of vertices adjacent to x. V_k (or $V_{\geq k}$, $V_{\leq k}$) is the set of k-vertices (or $(\geq k)$-vertices, $(\leq k)$-vertices, respectively). Let $d(x)$ be the degree of x. A *k-edge-coloring* of a graph G is a function $\phi : E(G) \mapsto \{1, \ldots, k\}$ such that any two adjacent edges receive different colors. The *edge chromatic number*, denoted by $\chi_e(G)$, of a graph G is the smallest integer k such that G has a k-edge-coloring. Vizing's Theorem [8] states that the edge chromatic number of a simple graph G is either Δ or $\Delta + 1$, where Δ denotes maximum vertex degree of G. A graph G is *class one* if $\chi_e(G) = \Delta$ and is *class two* otherwise. A class two graph G is *critical* if $\chi_e(G - e) < \chi_e(G)$ for each edge e of G. A critical graph G is *Δ-critical* if it has maximum degree Δ. In this paper, all graphs are simple and surfaces are compact, connected two manifolds without boundary. Embeddings considered in this paper are 2-cell embeddings. For a surface S, denote by c_S the *Euler characteristic* of the surface S. The best known lower bounds of size of critical graphs are listed in the following. Let G be a Δ critical graph with average degree q.

If $6 \leq \Delta \leq 8$, $\quad q \geq \frac{2}{3}(\Delta + 1)$ [4];

If $\Delta = 8$, $\quad q \geq 6 + \frac{1}{4}$ [10];

If $\Delta = 8$, $\quad q \geq 6\frac{1}{2}$ [5];

If $\Delta = 9$, $\quad q \geq 6\frac{4}{5}$ [10];

If $\Delta = 9$, $\quad q \geq 7\frac{1}{5}$ [5];

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$$\begin{array}{lll} \text{If } \Delta \geq 9, & q \geq 6\frac{2}{3} & [6]; \\ \text{If } \Delta = 10, & q \geq 7\frac{2}{5} & [10]; \\ \text{If } \Delta = 11, & q \geq 8 & [10]; \\ \text{If } 12 \leq \Delta \leq 14, & q \geq \frac{2}{3}(\Delta+1) & [10]. \end{array}$$

Some of those earlier results are improved in this paper:

$$\begin{array}{ll} \text{If } \Delta = 10, & q \geq 8; \\ \text{If } \Delta = 11, & q \geq 8\frac{1}{2}; \\ \text{If } \Delta = 12, & q \geq 9\frac{1}{5}. \end{array}$$

In Section 4, we obtain that if G is a simple graph with maximum degree Δ that is embeddable in a surface S of characteristic $c_S = -4$, or -5, or -6, then G is class one if $\Delta \geq 10$ or 11 or 12 respectively.

2. Adjacency lemmas

Throughout, let G denote a Δ-critical graph. In this section, we give some useful Adjacency Lemmas for critical graphs. The first one is the well-known Vizing's Adjacency Lemma [9], which will be abbreviated as VAL in this article.

VAL: If xy is an edge of a Δ-critical graph G, then x has at least $(\Delta - d(y) + 1)$ Δ-neighbors.

Zhang's Adjacency Condition (Zhang [11]): Let G be critical, $xy \in E(G)$ and $d(x) + d(y) = \Delta + 2$. Then every vertex at distance 2 from x or y has degree at least $\Delta - 1$, and has degree Δ if $d(x), d(y) < \Delta$.

Suppose that $G - e$ has a Δ-edge-coloring. Given two colors j and k, the subgraph of G induced by the edges colored either j or k, call it $G(j, k)$, has maximum degree two, and is thus the disjoint union of paths and cycles. A *bi-colored*(j, k)-path is a component of $G(j, k)$ which is a path. A vertex v *sees* color j if v is adjacent to an edge colored by j. Given a vertex v in G that sees j and does not see k, *swapping* (j, k) *along* v means swapping the coloring j and k along the (j, k)-bi-colored path starting at v. Denote by $P_{j,k}(v)_\phi$ the (j, k)-bi-colored path starting at v under edge coloring ϕ, or by $P_{j,k}(v)$ under current coloring if there is no confusion. Let ϕ be a Δ-edge-coloring of $G - e$.

Let $\phi(v)$ be the set of colors appearing on the edges adjacent to the vertex v. Recall that G is a Δ-critical graph and let $e = xw \in E(G)$. Facts 1 and 2 below are standard and easily obtained by swapping arguments.

Fact 1. If x sees i but not j and w sees j but not i, then the (i, j)-bi-colored path with an end at one of $\{x, w\}$ must end at another.

Fact 2. If z sees k but not j and w misses k, x misses j, then swapping (k, j) along z does not affect the colors of edges adjacent to x or w.

The following two Facts are due to Luo and Zhao [7].

Fact 3. Let $u \in N(x) \setminus \{w\}$, and edge xu be colored k. If w misses k, then u sees every color seen by only one of x, w.

Fact 4. Let u be a neighbor of x and v $(\neq x, w)$ be a neighbor of u. Assume that ux is colored k and uv is colored l. If k is missing at w and l is missing at either x, or w, then v sees every color seen by only one of x, w.

Lemma 2.1 (*Luo and Zhao [7]*). *Let G be a Δ-critical graph with $\Delta \geq 6$ and let x be a 4-vertex.*

(1) *If x is adjacent to four Δ-vertices and one of its neighbors is adjacent to three$(\leq \Delta - 2)$-vertices, then each of the remaining three neighbors of x is adjacent to only one $(\leq \Delta - 2)$-vertex, which is x;*
(2) *If x is adjacent to a $(\Delta - 1)$-vertex, then there are at least two Δ-vertices in $N(x)$ which are adjacent to at most two $(\leq \Delta - 2)$-vertices. Moreover, if x is adjacent to two $(\Delta - 1)$-vertices, then each of the two Δ-neighbors is adjacent to exactly one $(\leq \Delta - 2)$-vertex, which is x.*

Note that each color shows at either x or at w, or G has an edge Δ-coloring. Let $d(x) = d, d(w) = k(k \leq \Delta)$. For the purpose of convenience, without loss of generality, let the edges incident with x in G-xw be colored $1, \dots, d-1$ and denote the corresponding vertices by $x^1, \dots, x^{d-1}$ respectively, while those incident with w be colored $\Delta - k + 2, \dots, \Delta$ and denote the corresponding vertices by $w_{\Delta-k+2}, \dots, w_\Delta$ respectively (see Fig. 1). The proofs of several lemmas listed below will use the same labeling of $G - xw$ exhibited in Fig. 1. Because x and w are the critical edge vertices throughout, we use the terminology 'seen by x only' to mean seen by x but not by w, and 'seen by w only' to mean seen by w but not by x. In the Lemma statement below, b must have the value 1, but we use the label b to match its form to later results.

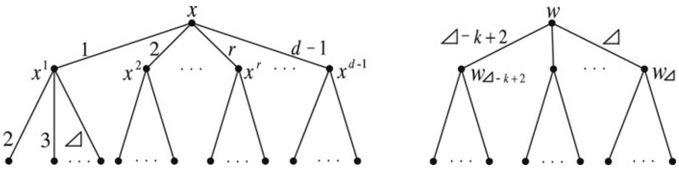

Fig. 1. Δ-edge coloring ϕ of $G - xw$ exhibited at $N(x) \cup N(w)$.

Lemma 2.2. *Let $d(x) = 5$ and $d(w) = \Delta$. If w_Δ misses a color $r \in \{2, 3, 4\}$, then $P_{b,r}(w_\Delta)$ must end at w and pass through w_r where b is seen by x only.*

Proof. Otherwise assume that $P_{b,r}(w_\Delta)$ does not end at w. We swap (b, r) along w_Δ since w_Δ sees b. Then w_Δ misses b which contradicts w_Δ seeing b by Fact 1 (because the $(1, \Delta)$ path starting at w has first edge ww_Δ and by Fact 1 ends at x). $\square$

Lemma 2.3. *For the Δ-coloring ϕ of $G - xw$ exhibited in Fig. 1, if $d(x) = 5$ and $d(w) = \Delta$, $P_{j,r}(w_\Delta)$ must end at x where $j \geq 5$, $r \in \{2, 3, 4\} = \phi(x) \cap \phi(w)$ if w_Δ misses r.*

Proof. Note that $d - 1 = 4$, $\Delta - k + 2 = 2$ here. Assume that $P_{j,r}(w_\Delta)$ does not end at x. Then swapping (r, j) along x does not affect the colors seen by w and x sees j but r, hence w_Δ must see r, a contradiction. $\square$

Lemma 2.4. *Let G be a Δ-critical graph with $\Delta \geq 10$ and x be a 5-vertex. If a Δ-neighbor of x, say w, is adjacent to four $(\leq \Delta-3)$-vertices, then, the remaining four neighbors of x are all Δ-vertices and none of them is adjacent to any $(\leq \Delta - 3)$-vertices except x.*

Proof. Note that, in this Lemma, $d(x) = 5$ and $d(w) = \Delta$ in Fig. 1. We have to show Claim A before giving the proof of (1).

Claim A. *Each neighbor $w_j (j \geq 5)$ of w must see each color seen by one of x, w only, that is, colors in $\{1, 5, 6, \ldots, \Delta\}$.*

This is immediate from Fact 3 (with the roles of x and w reversed).

(1) Three $(\leq \Delta - 3)$-neighbors of w except x must be $w_r (r = 2, 3, 4)$.

Proof of (1). By contradiction, we assume that there is a $(\leq \Delta - 3)$-neighbor w_i $(i \geq 5)$ of w, say w_Δ, which misses at least three colors. By Claim A, each w_j $(j \geq 5)$ must see each color seen by only one of x, w, that is, each color in $\{1\} \cup \{5, \ldots, \Delta\}$. Therefore, the three missing colors must be 2, 3 and 4.

Assume that, without loss of generality, w_Δ misses a color $r \in \{2, 3, 4\}$. Then we have Claims B and C, as explained below.

Claim B. *Each neighbor $w_j (j \neq \Delta)$ of w must see $r (= 2, 3, 4)$.*

Otherwise if w_j misses color r, by applying Lemma 2.2, both $P_{1,r}(w_\Delta)$ and $P_{1,r}(w_j)$ must pass w_r and end at w with a common edge ww_r, a contradiction. Hence, by combining the result from Claim A, we have $d(w_j) = \Delta (j \neq \Delta)$.

Claim C. *$w_r (r = 2, 3, 4)$ must see each color in $\{1\} \cup \{5, \ldots, \Delta\} \cup \{r\}$.*

Firstly, by Lemma 2.2, w_r sees color 1. Secondly, we assume that w_r misses a color $b \in \{5, \ldots, \Delta\}$. We swap $(1, b)$ along w_r which does not affect the colors of edges incident with x, w by Fact 2. Thus w_r does not see color 1 anymore which contradicts w_r seeing color 1. Hence $d(w_r) \geq \Delta - 2 (r = 2, 3, 4)$.

By Claims A–C, if there is a $(\leq \Delta - 3)$-neighbor w_i $(i \geq 5)$ of w, then each remaining neighbor of w is $(\geq \Delta - 2)$-vertex which implies that all three $w_r (r = 2, 3, 4)$ vertices must be the $(\leq \Delta - 3)$-vertices and the other $(\leq \Delta - 3)$-vertex is x. Hence, we complete the proof of (1).

(2) Claim that $d(x^1) = \Delta$ (see Fig. 1).

Proof of (2). (2-1) By Fact 3, x^1 must see each color seen by only one of x, w. That is, x sees every color in $\{1, 5, 6, \ldots, \Delta\}$.

(2-2) x^1 must see color $r (r = 2, 3, 4)$.

Suppose, to the contrary, that x^1 misses r. Since $d(w_r) \leq \Delta - 3$ and w_r sees r, w_r must miss a color seen only by one of x, w. Under the current assumption, we have the following claims: (2-2-1) and (2-2-2).

(2-2-1) Claim that w_r cannot miss a color $k \geq 5$.

Assume that w_r misses a color $k \geq 5$. Then the path $P_{r,k}(w_r)$ must end at x. Otherwise, swapping (r, k) along w_r implies that ww_r is colored by k which is not seen by x, so $d(w_r) \geq \Delta - 2$, a contradiction. Hence $P_{r,k}(w_r)$ ends at x and does not pass x^1. Swap (r, k) along w_r so that x sees k but not r. By Fact 3, x^1 sees r, a contradiction. Thus, w_r must see each color seen by w only.

(2-2-2) Claim that w_r cannot miss 1.

Suppose, to the contrary, that w_r misses 1. Then swapping $(\Delta - 1, 1)$ along w_r does not affect the colors of edges incident with x, w, then w_r misses $\Delta - 1$, a contradiction to the claim (2-2-1).

So w_r may only miss colors in $\{2, 3, 4\} \setminus \{r\}$ which contradicts that $d(w_r) \le \Delta - 3$. Hence x^1 sees 2, 3 and 4. Therefore, $d(x^1) = \Delta$ and we complete the proof of (2).

(3) Let y be a neighbor of x^1 and x^1y be colored by a color seen by only one of x, w. We claim that $d(y) = \Delta$.

Proof of (3). (3-1) y sees each color seen by only one of x, w by Fact 4.

(3-2) y sees each color r where $r = 2, 3, 4$.

Assume that y misses r. Without loss of generality, let x^1y be colored Δ. We consider w_r. Since $d(w_r) \le \Delta - 3$, then w_r must miss a color seen by only one of x, w. If w_r misses $k \ge 5$ and $k \ne \Delta$, then $P_{r,k}(w_r)$ must end at x. Otherwise, swapping (r, k) along w_r implies that ww_r is colored by k which is not seen by x. By applying the proof of (1), $d(w_r) \ge \Delta - 2$, a contradiction. Hence $P_{r,k}(w_r)$ ends at x and does not pass y. Swap (r, k) along w_r so that x sees k but not r. By Fact 4, y sees r, a contradiction. Therefore w_r sees each color in $\{5, \ldots, \Delta - 1\}$. Also w_r must see Δ. Otherwise, whether $P_{r,\Delta}(w_r)$ ends at x, or ends at y, swapping (r, Δ) along x brings us to the previous case. Moreover, w_r must see 1. Otherwise, swapping $(\Delta - 1, 1)$ along w_r does not affect the colors of edges incident with x, w. Then, w_r misses $\Delta - 1$, a contradiction again. So w_r may only miss colors in $\{2, 3, 4\} \setminus \{r\}$, but then it contradicts that $d(w_r) \le \Delta - 3$. Hence y sees each color $r \in \{2, 3, 4\}$. Therefore, $d(y) = \Delta$.

(4) Let y_r be a neighbor of x^1 and x^1y_r be colored r where $r = 2, 3, 4$. We claim that $d(y_r) \ge \Delta - 2$.

Proof of (4). Without loss of generality, we have to show that $d(y_2) \ge \Delta - 2$. Since $d(w_2) \le \Delta - 3$, w_2 must miss a color seen by one of x, w only.

(4-1) If w_2 misses a color $i \ge 5$, then $d(y_2) \ge \Delta - 2$.

Without loss of generality, assume that w_2 misses Δ. We claim that path $P_{2,\Delta}(w_2)$ must end at x. Otherwise, swapping $(2, \Delta)$ along w_2 shows that $d(w_2) \ge \Delta - 2$ since ww_2 is colored Δ now, a contradiction.

(4-1-1) y_2 must see Δ.

Assume that y_2 misses Δ. Then $P_{2,\Delta}(y_2)$ must end at x. Otherwise, swapping $(2, \Delta)$ along y_2 indicates that $d(y_2) = \Delta$ by (3). But then both paths $P_{2,\Delta}(w_2)$ and $P_{2,\Delta}(y_2)$ end at x and have a common edge xx^2, a contradiction.

(4-1-2) y_2 sees 1.

By Fact 1, $P_{1,\Delta}(x)$ ends at w. If y_2 misses 1, then $P_{\Delta,1}(y_2)$ does not end at x. Swapping $(\Delta, 1)$ along y_2 does not affect the colors of edges incident with x, w. So after the swapping, y_2 sees 1 but not Δ which contradicts the result that y_2 must see Δ.

(4-1-3) y_2 sees each color seen by w and not by x.

Assume that y_2 misses a color seen by w only, say 5. Note that $P_{1,5}(x)$ ends at w, we swap $(1, 5)$ along y_2 and the colors of edges incident with x, w have not been affected. Hence y_2 sees 5 but not 1 which contradicts the fact that y_2 must see 1.

From the discussion above, y_2 may only miss colors 3 and 4, so $d(y_2) \ge \Delta - 2$.

(4-2) If w_2 misses a color seen by x and not by w (which is 1 here), then $d(y_2) \ge \Delta - 2$.

We swap $(\Delta, 1)$ along w_2. Then w_2 sees 1 but not Δ which brings us to the previous case.

Hence, by combining the results in (3) and (4), each neighbor of x^1 other than x is a $(\ge \Delta - 2)$-vertex.

(5) Claim that $d(x^2) = \Delta$ (see Fig. 1). If z is a neighbor of x^2 other than x, then either $d(z) = \Delta$ or $d(z) \ge \Delta - 2$.

Proof of (5). w_2 misses a color seen by either x or w since $d(w_2) \le \Delta - 3$. We can assume that w_2 misses 1. Otherwise, w_2 misses a color $i \ge 5$, say Δ. Then swapping $(1, \Delta)$ along w_2 does not affect the colors of edges incident with x, w. After the swapping, w_2 misses 1, and we re-color ww_2 by 1 since w_2 misses 1 under the current coloring. Thus, color 2 is seen by x and not by w. Now x^2 plays the same role as x^1 does before the swapping. Hence we have our required results.

(6) Similarly, x^3, x^4 are Δ-vertices and adjacent to all $(\ge \Delta - 2)$-vertices by using a proof similar to that in (5) since $d(w_3), d(w_4) \le \Delta - 3$. □

Lemma 2.5. *Let G be a Δ-critical graph with $\Delta \ge 10$ and x be a 5-vertex with a $(\Delta - 1)$-neighbor w.*

(i) *If w is adjacent to two $(\le \Delta - 2)$ vertices other than x, the remaining four neighbors of x are all Δ-vertices and each of them is adjacent to all $(\ge \Delta - 1)$-vertices except x.*
(ii) *If w is adjacent to one $(\le \Delta - 2)$ vertices other than x, then there are three $(\ge \Delta - 1)$-neighbors y of x including at least one Δ-neighbor satisfying the following situations: if y is a Δ-vertex, then it is adjacent to at most two $(\le \Delta - 1)$-vertices; if y is a $(\Delta - 1)$-vertex, then it is adjacent to only one $(\le \Delta - 1)$-vertex which is x.*

Proof. Let ϕ be a Δ-edge-coloring of $G - xw$. Without loss of generality, the edges incident with x in G-xw are colored $1, \ldots, 4$ and denote the corresponding vertices by $x^1, \ldots, x^4$, while those incident with w are colored $3, \ldots, \Delta$ respectively and denote the corresponding vertices by $w_3, \ldots, w_\Delta$ respectively (see Fig. 2). Please note that $\phi(x) \cap \phi(w) = \{3, 4\}$.

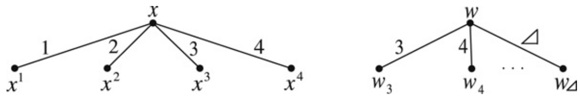

Fig. 2. Δ-edge coloring ϕ of G-xw exhibited at $N(x) \cup N(w)$ in Lemma 2.5.

Proof of (i). By a proof similar to that in Lemma 2.4(1), each neighbor $w_j (j \geq 5)$ of w sees every color seen by either x or w. If there is one w_j $(j \geq 5)$ missing both colors 3 and 4, then each of the remaining neighbors of w must have a degree of Δ which contradicts the assumption that w is adjacent to two $(\leq \Delta - 2)$ vertices other than x. So there is no $w_j (j \geq 5)$ missing both 3 and 4. Hence the two $(\leq \Delta - 2)$-neighbors of w must be w_3 and w_4.

(1) Claim that $d(x^1) = d(x^2) = \Delta$.

(1-1) By Fact 3, each of $\{x^1, x^2\}$ sees every color seen by either x or w since 1 and 2 are seen by x and not by w.

(1-2) x^1 and x^2 must see color $r (r = 3, 4)$.

Suppose, to the contrary, without loss of generality, that x^1 misses r. Since $d(w_r) \leq \Delta - 2$, w_r must miss a color seen by either x or w. We have to show a contradiction by (1-2-1) and (1-2-2) as explained below.

(1-2-1) $w_r (r = 3, 4)$ cannot miss a color $k \geq 5$.

Suppose, to the contrary, that w_r misses a color $k \geq 5$. Then the path $P_{r,k}(w_r)$ must end at x. Otherwise, swapping (r, k) along w_r indicates that ww_r is colored by k which is not seen by x. So $d(w_r) \geq \Delta - 1$, a contradiction. Hence $P_{r,k}(w_r)$ ends at x and does not pass x^1. Swap (r, k) along w_r so that x sees k but not r. On the other hand, however, by Fact 3, x^1 must see r, a contradiction. Thus, w_r must see each color seen by w and not by x.

(1-2-2) w_r cannot miss color $b \in \{1, 2\}$.

Suppose, to the contrary, that w_r misses b. Swapping $(\Delta - 1, b)$ along w_r does not affect the colors of edges incident with x, w, then w_r misses $\Delta - 1$, a contradiction. Thus, the missed colors by w_r may be a color in $\{3, 4\} \setminus \{r\}$ which contradicts the fact that $d(w_r) \leq \Delta - 2$.

Contradictions from (1-2-1) and (1-2-2) make our assumption fail. Hence x^1 sees $r \in \{3, 4\}$. Therefore, $d(x^1) = \Delta$. Similarly, $d(x^2) = \Delta$.

(2) Let $y \in N(x^1) \setminus \{x\}$. By a proof similar to that in Lemma 2.4, either $d(y) = \Delta$ or y may miss only one color in $\{3, 4\}$, so $d(y) \geq \Delta - 1$. Similarly, if $z \in N(x^2) \setminus \{x\}$, then $d(z) \geq \Delta - 1$ since 2 is seen by x and not by w.

(3) $d(z) \geq \Delta - 1$ if $z \in N(x^r) \setminus \{x\}$ where $r = 3, 4$.

Let $z \in N(x^3) \setminus \{x\}$. We consider w_3. We may assume that w_3 misses 1. Since $d(w_3) \leq \Delta - 2$, then w_3 must miss a color seen by either x or w. If w_3 misses a color no smaller than 5, say 5 but sees 1, then we swap $(1, 5)$ along w_3, so that w_3 misses 1 and colors of edges incident with x, w have not been affected. We re-color ww_3 by 1. Now 3 is seen by x and not by w, and note that xx^3 is colored 3. So x^3 plays the same role as x^1 did before the re-coloring. Thus we have our required results. Similarly, for the case of x^4, because $d(w_4) \leq \Delta - 2$, if $z \in N(x^4) \setminus \{x\}$, then $d(z) \geq \Delta - 1$.

Proof of (ii). The proof of (ii) consists of two parts: Part A and Part B.

Part A Assume that the one $(\leq \Delta - 2)$-neighbor of w besides x is w_j where $j \geq 5$.

Without loss of generality, assume that $j = \Delta$. Then we claim that the two colors missed by w_Δ must be 3 and 4, and each remaining neighbor of w sees all colors by a proof similar to that in Lemma 2.4.

(A-1) $d(x^i) = \Delta (i = 1, 2)$, and x^i is adjacent to all Δ-vertices other than x.

Without loss of generality, we show that $d(x^1) = \Delta$ since x^2 plays the same role as x^1. Firstly, x^1 sees each color seen by either x or w by Facts 3 and 4. Secondly, x^1 must see 3 and 4. Otherwise, without loss of generality, assume that x^1 misses 3. Please note that $P_{\Delta,3}(w_\Delta)$ must end at x by Lemma 2.3. If we swap $(\Delta, 3)$ along w_Δ, in the new coloring x sees Δ but not 3; in this case x^1 sees 3 since it is seen by w only. Similarly, x^1 sees color 4 since w_Δ misses 4 also.

(A-1-1) Let y be a neighbor of $x^i (i = 1, 2)$ with x^iy colored by a color seen by only one of x, w, then $d(y) = \Delta$.

Proof of (A-1-1). Clearly, y sees each color seen by only one of x, w. Next, we show that y sees 3 and 4. Assume, without loss of generality, y misses 3. Then $P_{\Delta,3}(w_\Delta)$ does not pass y. Swapping $(\Delta, 3)$ along w_Δ indicates that x sees Δ but not 3, so y must see 3, a contradiction.

(A-1-2) Let z be a neighbor of $x^i (i = 1, 2)$ with x^iz colored by a color seen by both x and w; then $d(z) = \Delta$.

Proof of (A-1-2). Without loss of generality, let x^iz be colored 3. Firstly, we show that z sees each color $q (\geq 5)$. Otherwise, assume that z misses $q (\geq 5)$. If $q \neq \Delta$, note that $P_{\Delta,3}(x)$ ends at w_Δ, swapping $(3, \Delta)$ along x shows that x sees Δ but not 3. Thus, z plays the same role as y in (A-1-1) since color 3 is seen by w only. Hence $d(z) = \Delta$, a contradiction. If $q = \Delta$, then swapping $(3, \Delta)$ along z does not affect colors seen by x and w. But then z sees Δ but not 3 which implies that z is playing the same role as y in (A-1-1), so $d(z) = \Delta$, a contradiction.

Secondly, z sees b where $b = 1, 2$. Otherwise, swapping (Δ, b) along z does not affect the colors seen by x, w since $P_{b,3}(w_\Delta)$ ends at w by applying Lemma 2.2 here. Thus, z sees b but not Δ, a contradiction to the fact that z must see each color $q \geq 5$. Hence z sees each color seen by either x or w.

Lastly, z sees color 3 and 4. Note that x^iz is colored 3. We have to show that z sees 4. Since $P_{\Delta,4}(w_\Delta)$ ends at x, it does not pass z. Swapping $(4, \Delta)$ along x shows that x sees Δ but not 4. So z must see 4 since 4 is seen by w only.

(A-2) $d(x^j) = \Delta (j = 3, 4)$ and x^j is adjacent to all Δ-vertices except x.

Without loss of generality, we give detailed discussion for x^3 only.

(A-2-1) $d(x^3) = \Delta$.

Firstly, x^3 sees $b(b = 1, 2)$. Otherwise, swapping $(3, b)$ along x^3 does not affect colors seen by w and x since $P_{3,b}(w_\Delta)$ end at w by applying Lemma 2.2. Now x^3 plays the same role as x^1 did before the swapping, so $d(x^3) = \Delta$, a contradiction.

Secondly, x^3 sees $q(\geq 5, \neq \Delta)$. Otherwise if $q \neq \Delta$, we re-color xx^3 by the missing color q, so x sees q but not 3. Thus, $P_{\Delta,3}(w_\Delta)$ does not end at x under the current coloring, a contradiction.

Thirdly, x^3 sees Δ. Otherwise, $P_{\Delta,3}(w_\Delta)$ cannot end at x, a contradiction. Hence x^3 sees every color seen by either x or w.

Lastly, x^3 must see 4. Otherwise, swapping $(1, 4)$ along x^3 does not affect the colors seen by x, w since $P_{1,4}(w_\Delta)$ must end at w. After the swapping, x^3 misses 1 which violates that x^1 must see 1. Be aware that xx^3 is colored 3, so $d(x^3) = \Delta$. Similarly $d(x^4) = \Delta$.

(A-2-2) Let y be a neighbor of $x^j (j = 3, 4)$ with x^jy colored by a color $q \geq 5$ seen by w only, then $d(y) = \Delta$.

Without loss of generality, let $j = 3$. Firstly, y sees 3 since $P_{q,3}(w_\Delta)(q \geq 5)$ must end at x by applying Lemma 2.3. Secondly, y sees $i \geq 5$. Otherwise, swapping $(3, i)$ along y does not affect colors seen by x, w but then y misses 3, a contradiction. Thirdly, y sees $b(= 1, 2)$, or else swapping $(3, b)$ along y does not affect the colors seen by x, w since $P_{b,3}(w_\Delta)$ ends at w. But then y misses 3, again we have a contradiction. Lastly, y sees 4. Otherwise, swapping $(\Delta, 4)$ along y does not affect the colors seen by x, w which means that y misses Δ, a contradiction again.

(A-2-3) Let z be a neighbor of $x^j (j = 3, 4)$ with x^jz colored by a color $b(= 1, 2)$ seen by x but not by w, then $d(z) = \Delta$.

First z sees $q \geq 5$, or else swapping (b, q) along z does not affect colors seen by x, w because of the Fact 2. Then x^jz is colored by $q \geq 5$, so $d(z) = \Delta$ by (A-2-2), a contradiction. Second z sees 3 and 4. Assume that z misses 3, then swapping $(\Delta, 3)$ along z does not affect colors seen by x, w because $P_{3,\Delta}(x)$ ends at w_Δ. But then z misses Δ which violates that z must see Δ. Third z sees $c \in \{1, 2\} \setminus \{b\}$. Otherwise, swapping (Δ, c) along z does not affect colors seen by x, w due to Fact 1, but then z misses Δ, a contradiction again.

(A-2-4) Let u be a neighbor of $x^j (j = 3, 4)$ with x^jz colored by a color $k \in \{3, 4\} \setminus \{j\}$ seen by both x and w; then $d(u) = \Delta$.

Without loss of generality, let $j = 3$, then $k = 4$. First u sees Δ. Otherwise, swapping $(4, \Delta)$ along u does not affect the colors seen by x, w by Lemma 2.3. Now edge x^3u is colored by Δ. By (A-2-2), $d(u) = \Delta$. Second u sees each color seen by either x or w. If u misses a color, say q which seen by either x or w, then swapping (Δ, q) along u shows that u misses Δ, a contradiction. Third u sees 3. Otherwise, swapping $(\Delta, 3)$ along u does not affected colors seen by x, w since $P_{\Delta,3}(w_\Delta)$ must end at x. But in the new coloring, u misses Δ, a contradiction.

Hence, we finish our proof of Part A.

Part B Assume that the one $(\leq \Delta - 2)$-neighbor of w other than x is w_j where $j = 3$, or 4.

Without loss of generality, assume $d(w_3) \leq \Delta - 2$, then x^1, x^2 may miss color 4. Let $y \in N(x^i)(i = 1, 2)$. If x^iy $(i = 1, 2)$ is colored by a color seen only by one of x, w, then y must see all colors other than 4. If x^iy is colored by 3, by a proof similar to that of (4) in Lemma 2.4, y must see all colors except 4. So $d(y) \geq \Delta - 1$. Furthermore, if $x^i (i = 1, 2)$ is a Δ-vertex, x^i sees color 4. Let $z \in N(x^i)$ and x^iz be colored by 4, then z may be a $(\leq \Delta - 2)$-vertex. So x^i is adjacent to one $(\leq \Delta - 2)$-vertex which is x if x^i is a $(\Delta - 1)$-neighbor of x, and x^i is adjacent to at most two $(\leq \Delta - 2)$-vertices if x^i is a Δ-neighbor of x.

By combining Part A and Part B, we have our results. Hence we finish our proof of (ii). □

For a special case of that x is a 4-vertex, the following result will be used in the proof of Theorem 3.1.

Corollary 2.6. *Let x be a 4-vertex in a Δ-critical graph G, and suppose that x has a $(\Delta - 1)$-neighbor w. If w is adjacent to one $(\leq \Delta - 1)$-vertex other than x, then all the remaining three neighbors of x are Δ-vertices and each of them is adjacent to all $(\geq \Delta - 1)$-vertices except x.*

Since the proof is similar to that of the previous Lemma, we omit it here.

Lemma 2.7. *Let x be a 5-vertex in a Δ-critical graph G and suppose that x has a $(\Delta - 2)$-neighbor w.*

(i) *If w is adjacent to one $(\leq \Delta - 2)$-vertex other than x, then all the remaining four neighbors of x are Δ-vertices and each of them is adjacent to $(\geq \Delta - 1)$-vertices except x.*
(ii) *If w is adjacent to only one $(\leq \Delta - 2)$-vertex which is x, then there are three $(\geq \Delta - 1)$-neighbors of x including at least two Δ-neighbors y satisfying the following situations: if it is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta - 2)$-vertices; if it is a $(\Delta - 1)$-vertex, then it is adjacent to one $(\leq \Delta - 2)$-vertex which is x.*

Proof. We use the same the notation for edges and vertices of $N(x) \cup N(w)$ of G-xw as in Lemma 2.5. Be aware that $\phi(x) \cap \phi(w) = \{4\}$.

Proof. By a proof similar to that of (1) in Lemma 2.4, each neighbor $w_j (j \geq 5)$ sees each color seen by either x or w. Thus the $(\leq \Delta - 2)$-neighbor of w must be w_4. Since $d(w_4) \leq \Delta - 2$, the vertex w_4 must miss a color seen by either x or w. By using an argument similar to that in Lemma 2.4, for each vertex $z \in N(x) \setminus \{w\}$, z is adjacent to all $(\geq \Delta - 1)$-vertices except x.

Proof of (ii). Applying the same arguments to that in Lemma 2.5, x^1, x^2 and x^3 are three $(\geq \Delta - 1)$-neighbors of x (each of them may miss only color 4). Be aware that $d_\Delta(x) \geq 3$. So two vertices of $\{x^1, x^2, x^3\}$ must be Δ-vertices.

By using the same method as in Lemma 2.5(ii), we have our results. Since the proof is similar to that of Lemma 2.5(ii), we omit it. □

Similarly, we have results on 6-vertices.

If $x \in V(G)$, we denote the degree of the neighbors of x as $\delta_1(x) \leq \delta_2(x) \leq \cdots \leq \delta_{d(x)}(x)$.

Lemma 2.8. *Let x be a 6-vertex in a Δ-critical graph G and w be a $\delta_1(x)$-neighbor of x where $\delta_1(x) = \Delta - 2$, or $\Delta - 1$. Then we have following:*

(i) *Case one. $\delta_1(x) = \Delta - 2$.*

(i-1) *If w is adjacent to three $(\leq \Delta - 2)$-vertices, then each of the remaining five neighbors of x is Δ-vertex and is adjacent to all $(\geq \Delta - 2)$-vertices except x.*

(i-2) *If w is adjacent to two $(\leq \Delta - 2)$-vertices, then there are four $(\geq \Delta - 1)$-neighbors of x including at least two Δ-neighbors y satisfying: if y is a Δ-vertex, then it is adjacent to at most two $(\leq \Delta - 2)$-vertices; if y is a $(\Delta - 1)$-vertex, then it is adjacent to one $(\leq \Delta - 2)$-vertex which is x.*

(i-3) *If (i-1) and (i-2) do not happen, each $\Delta - 2$-neighbor of x is adjacent to one $(\leq \Delta - 2)$-vertex which is x, and each Δ-neighbor of x is adjacent to at most three $(\leq \Delta - 2)$-vertices.*

(ii) *Case two. $\delta_1(x) = \Delta - 1$.*

(ii-1) *If $(\Delta - 1)$-neighbor w is adjacent to four $(\leq \Delta - 3)$-vertices, then each of the remaining five neighbors of x is Δ-vertex and is adjacent to all $(\geq \Delta - 2)$-vertices except x.*

(ii-2) *Or each $(\Delta - 1)$-neighbor of w is adjacent to at most three $(\leq \Delta - 3)$-vertices.*

We use the same way to set up notations and labels for vertices and edges of $N(x) \cup N(w)$ of $G - xw$ as in Lemma 2.4. Be aware that colors 1, 2 and 3 play same role here. To avoid repetition, we omit the proof.

3. Main results

Theorem 3.1. *Let G be a Δ-critical graph with $\Delta \geq 10$. Then $|E(G)| \geq \frac{|V(G)|}{2} q$ where*

$$q = \begin{cases} 8 & \text{if } \Delta = 10 \\ 8.5 & \text{if } \Delta = 11 \\ 9.2 & \text{if } \Delta = 12. \end{cases}$$

Proof. Suppose to the contrary, the theorem is not true. Then

$$\sum_{x \in V} (d(x) - q) < 0.$$

We have to use Charge-Discharge method to get a contradiction. For each vertex $x \in V(G)$, we call $c(x) = d(x) - q$ the *initial charge* of the vertex x and will assign a new charge to each vertex x according to the following rules.

(R1) Let x be a 2-vertex and $u, v \in N(x)$. x receives $d(y) - q$ from each adjacent Δ-vertex y and each $z \in N(u) \setminus \{x, v\}$ sends $\frac{d(z)-q}{\Delta}$ to x via u and each $z \in N(v) \setminus \{x, u\}$ sends $\frac{d(z)-q}{\Delta}$ to x via v. Note that any Δ-vertex adjacent to both u and v sends $2 \times \frac{d(z)-q}{\Delta}$ to x in total.

(R2) Let x be a $(\leq q)$-vertex.

(R2.1) If x is a 7-vertex and $\Delta = 10$. Then x receives $\frac{1}{15}$ from each adjacent 9-vertex.

(R2.2) Otherwise, x receives $\frac{d(y)-q}{j}$ from each adjacent $(> q)$-vertex y with $d_{<q}(y) = j$.

Let $B = \frac{q - \lfloor q \rfloor}{4}$.

(R3) Let x be a $\lfloor q \rfloor$-vertex. Note that $c(x) \leq 0$, then x receives B from each $(\geq \Delta - 1)$-neighbor if $\Delta = 11, 12$, and receives $\frac{\Delta - 2 - q}{j}$ from each adjacent $(\Delta - 2)$-vertex y with $d_{\lfloor q \rfloor}(y) = j$.

Let $c'(x)$ be the new charge of each vertex.

(I) Claim that $c'(x) > 0$ if $d(x) = 2$.

Let $u, v \in V_\Delta \cap N(x)$. By Zhang's Adjacency Condition, each of u, v is adjacent to at least $(\Delta - 2)\Delta$-vertices different from u, v. Therefore, by (R1), $c'(x) \geq c(x) + 2 \times (\Delta - q) + 2 \times (\Delta - 2) \times \frac{\Delta - q}{\Delta} > 0$.

Denote by $\delta(x)$ the minimum degree of vertices adjacent to x.

(II) Claim that $c'(x) \geq 0$ if $d(x) + \delta(x) = \Delta + 2$.

Let y be a vertex adjacent to x with $d(x) + d(y) = \Delta + 2$. Assume that x is a d-vertex with $3 \leq d \leq \lfloor q \rfloor$. By Zhang's Adjacency Condition, $|N(x) \cap V_\Delta| = d - 1$ and each vertex in $N(N(x)) \setminus \{x, y\}$ has degree $\geq \Delta - 1$. Considering x, y may share some ($\geq \Delta - 1$)-vertices, so x receives $\Delta - q$ or $\frac{\Delta-q}{2}$ or B from each adjacent Δ-vertex and $\max\{d(y) - q, 0\}$ from y. So by (R2) and (R3),

$$c'(x) \geq \begin{cases} (d(x) - q) + (d(x) - 1)(\Delta - q) + (\Delta - 1 - q) > 0 & \text{if } d(x) = 3, 4, \\ (d(x) - q) + (d(x) - 1)\dfrac{\Delta - q}{2} > 0 & \text{if } 5 \leq d(x) < \lfloor q \rfloor, \\ (d(x) - q) + (d(x) - 1) \times B > 0 & \text{if } d(x) = \lfloor q \rfloor. \end{cases}$$

Assume that x is a ($> \lfloor q \rfloor$)-vertex. Then x sends out $d(x) - q$ to its adjacent vertex y, so $c'(x) \geq 0$. From now on, we consider $d(x) + \delta(x) \geq \Delta + 3$.

(III) Claim that $c'(x) > 0$ if $d(x) = 3$.

By (II), assume $\delta(x) = \Delta$, by Lemma 2.3 in [6], there are $two\Delta$-vertices in $N(x)$, each of them is adjacent to at least $(\Delta - 1)(\geq \Delta - 1)$-vertices. Hence, by (R2) and VAL, $c'(x) \geq c(x) + 2 \times (\Delta - q) + \frac{\Delta-q}{2} > 0$ for $\Delta = 10, 11, 12$.

(IV) Claim that $c'(x) > 0$ if $d(x) = 4$.

If $\delta(x) = \Delta - 1$, x is adjacent either $two\Delta$-vertices and $two(\Delta - 1)$-vertices, or $three\Delta$-vertices and $one(\Delta - 1)$-vertices. For the former case, by Lemma 2.1(3) and (R2), x receives $2 \times (\Delta - q)$ from its adjacent Δ-vertices, $c'(x) = 4 - q + 2 \times (\Delta - q) \geq 0$. For the latter case, if the $(\Delta - 1)$-neighbor of x is adjacent to at least two ($\leq \Delta - 2$)-vertices, then there are three Δ-neighbors of x which are adjacent to at most two ($\leq \Delta - 2$)-vertices, thus x receives $3\frac{\Delta-q}{2}$ from those Δ-neighbors and $\frac{\Delta-1-q}{2}$ from the $(\Delta - 1)$-neighbor. So $c'(x) \geq c(x) + 3\frac{\Delta-q}{2} + \frac{\Delta-1-q}{2}$. It is straightforward to check that $c'(x) > 0$. If the $(\Delta - 1)$-neighbor of x is adjacent to one ($\leq \Delta - 2$)-vertex which is x, two Δ-neighbors of x are adjacent to at most two ($\leq \Delta - 2$)-vertices by Lemma 2.1(3), so x receives $2\frac{\Delta-q}{2}$ from those two Δ-neighbors, $(\Delta - 1 - q)$ from the $(\Delta - 1)$-neighbor and $\frac{\Delta-q}{3}$ from the rest Δ-neighbor. It is straightforward to check that $c'(x) \geq c(x) + 2\frac{\Delta-q}{2} + (\Delta - 1) + \frac{\Delta-q}{3} > 0$.

If $\delta(x) = \Delta$, by Corollary 2.6, either there is $one\Delta$-neighbor of x which is adjacent to three ($\leq \Delta - 2$)-vertices including x, and each of rest Δ-neighbors is adjacent to one ($\leq \Delta - 2$)-vertex which is x, or each of Δ-neighbors of x is adjacent to at most two ($\leq \Delta - 2$)-vertices. For the former case, x receives at least $\frac{\Delta-q}{3} + 3 \times (\Delta - q)$ and it is easy to check that $c'(x) > 0$. For the latter case, x receives at least $4 \times \frac{\Delta-q}{2}$ from its adjacent vertices for $\Delta = 10, 11, 12$ respectively. Hence, $c'(x) \geq (4 - q) + 4\frac{\Delta-q}{2} > 0$.

(V) Claim that $c'(x) \geq 0$ if $d(x) = 5$.

Case 1 $\delta(x) = \Delta - 2$. Let w be the $(\Delta - 2)$-neighbor of x. Note that $d_\Delta(x) \geq 3$. If w is adjacent to two ($\leq \Delta - 1$)-vertices, then there are four Δ-neighbors of x and each of them is adjacent to all ($\geq \Delta - 1$)-vertices except x, so x receives $4(\Delta - q)$ from those Δ-neighbors, hence $c'(x) \geq (5 - q) + 4(\Delta - q) > 0$. If w is adjacent to one ($\leq \Delta - 1$)-vertex which is x, then by Lemma 2.7, there are three ($\geq \Delta - 1$)-neighbors of x including at least two Δ-neighbors and each of them is adjacent to at most two ($\leq \Delta - 1$)-vertices. x receives at least $3\frac{\Delta-q}{2}$ from those three Δ-neighbors. So $c'(x) \geq (5 - q) + 3\frac{\Delta-q}{2} \geq 0$.

Case 2 $\delta(x) = \Delta - 1$.

Subcase (a) If w is adjacent to three ($\leq \Delta - 2$)-vertices, then x has four Δ-neighbors and each of them is adjacent to all ($\geq \Delta - 2$)-vertices except x, so x receives $4(\Delta - q)$, and $c'(x) > 0$.

Subcase (b) If w is adjacent to two ($\leq \Delta - 2$)-vertices, then x has three ($\geq \Delta - 1$)-neighbors including one Δ-neighbor and each of them is adjacent to all ($\geq \Delta - 2$)-vertices except x. By Lemma 2.7(ii), x receives $(\Delta - 1 - q)$ from each of those $(\Delta - 1)$-neighbors and $\frac{\Delta-q}{2}$ from each of those Δ-neighbors. Hence x receives either $2(\Delta - 1 - q) + \frac{\Delta-q}{2}$ from those three neighbors if there is one Δ-vertex among them, or $(\Delta - 1 - q) + 2\frac{\Delta-q}{2}$ from those three neighbors if there are two Δ-vertices among them. Thus x receives at least $\min\{2(\Delta - 1 - q) + \frac{\Delta-q}{2}, (\Delta - 1 - q) + 2\frac{\Delta-q}{2}\}$. It is straightforward to check that $c'(x) \geq 0$.

Subcase (c) Each $(\Delta - 1)$-neighbor of x is adjacent to one ($\leq \Delta - 2$)-vertex which is x. By VAL and (R2), x receives $d_\Delta(x)\frac{\Delta-q}{4}$ from Δ-neighbors and $(d(x) - d_\Delta(x))(\Delta - 1 - q)$ from $(\Delta - 1)$-neighbors, then $c'(x) \geq (5 - q) + d_\Delta(x)\frac{\Delta-q}{4} + (d(x) - d_\Delta(x))(\Delta - 1 - q)$. It is straightforward to check that $c'(x) \geq 0$ for $\Delta = 10, 11, 12$ respectively where $d_\Delta(x) \geq 2$.

Case 3 $\delta(x) = \Delta$.

By Lemma 2.4, if one of Δ-neighbors of x is adjacent to *four* ($\leq \Delta - 3$)-vertices, then, the rest four neighbors of x are all Δ-vertices and each of them is not adjacent to any ($\leq \Delta - 3$)-vertices except x. In this case, x receives at least $(5 - q) + 4(\Delta - q)$ and $c'(x) > 0$. Otherwise each Δ-neighbor of x is adjacent to at most *three*($\leq \Delta - 3$)-vertices, so each Δ-neighbor gives $x\frac{\Delta-q}{3}$ by VAL, (R2) and (R3). It is directly to check that $c'(x) \geq (5 - q) + 5\frac{\Delta-q}{3} > 0$.

(VI) Claim that $c'(x) \geq 0$ if $d(x) = 6$.

If $\delta(x) = \Delta - 3$, x is adjacent to at least *four* Δ-vertices, and by Lemma 2.3. There are at least *three*Δ-vertices $z \in N(x)$ such that $d_{\leq \Delta-2}(z) \leq 3$. By (R2) and (R3), $c'(x) \geq (6 - q) + 3 \times \frac{\Delta-q}{3} + \frac{\Delta-q}{5} \geq 0$ for $\Delta = 10, 11, 12$ respectively.

If $\delta(x) = \Delta - 2$, let w be the $(\Delta - 2)$-neighbor of x. Be aware that $d_\Delta(x) \geq 3$. By Lemma 2.8, we consider the following three cases.

(a) If $d_{\leq \Delta-2}(w) \leq 3$, by Lemma 2.8(i), then three Δ-neighbors of x are all adjacent to ($\geq \Delta - 2$)-vertices except x. So x receives at least $3 \times (\Delta - q)(\geq -(6 - q))$. Hence, $c'(x) \geq 0$ for $\Delta = 10, 11, 12$ respectively.

(b) If $d_{\leq\Delta-2}(w) = 2$, then each of rest vertices $z \in N(x)$ has $d_{\Delta-2}(z) \leq 2$. Each Δ-neighbor z sends at least $\frac{\Delta-q}{2}$ to x, so x receives at least $3 \times (\frac{\Delta-q}{2}) + 3 \times (\frac{\Delta-q-2}{2}) > -(6-q)$ for $\Delta = 10, 11, 12$ respectively. Hence, $c'(x) \geq 0$.

(c) Each $(\Delta-2)$-vertex $w \in N(x)$ has $d_{\leq\Delta-2}(w) = 1$ and each Δ-vertex $z \in N(x)$ has $d_{\leq\Delta-2}(z) \leq 3$. By Lemma 2.8 x receives at least $3 \times \frac{\Delta-q}{3} + 3 \times \frac{\Delta-2-q}{3} > -(6-q)$ for $\Delta = 10, 11, 12$ respectively. Hence, $c'(x) \geq 0$.

If $\delta(x) = \Delta - 1$ or Δ, then x is either adjacent to *two* Δ-vertices and *four* $(\Delta-1)$-vertices, or at least *three* Δ-vertices. By Lemma 2.8, x receives at least

$$l \geq \begin{cases} 5(\Delta-q) \text{ or } 2 \times \dfrac{2}{5} + 4 \times \dfrac{1}{3} & \text{if } \delta(x) = \Delta - 1,\ \Delta = 10 \\ 6 \times \dfrac{2}{5} & \text{if } \delta(x) = \Delta,\ \Delta = 10 \\ 2\dfrac{\Delta-q}{5} + 4\dfrac{\Delta-1-q}{4} & \text{if } d_{\Delta}(x) = 2,\ \Delta = 11, 12 \\ 3\dfrac{\Delta-q}{5} + 3\dfrac{\Delta-1-q}{4} & \text{if } d_{\Delta}(x) \geq 3. \end{cases}$$

It is direct to check that $c'(x) \geq (6-q) + l \geq 0$.

(VII) Claim $c'(x) \geq 0$ if $d(x) = 7$.

Note that x is either adjacent to *two* Δ-vertices and *five* $(\Delta-1)$-vertices, or at least *three* Δ-vertices and *four* $(\geq \Delta-2)$-vertices. Then x receives at least

$$\begin{cases} 2\dfrac{2}{6} + 5 \times \dfrac{1}{15} \text{ or } 3 \times \dfrac{1}{3} & \text{if } \Delta = 10 \\ 2\dfrac{\Delta-q}{6} + 5\dfrac{\Delta-1-q}{5} \text{ or } 3\dfrac{\Delta-q}{6} + 4\dfrac{\Delta-2-q}{4}, & \text{if } \Delta = 11, 12. \end{cases}$$

It is direct to check that $c'(x) \geq 0$.

(VIII) Claim that $c'(x) \geq 0$ if $d(x) = 8$.

If $\Delta = 10$, $c'(x) = c(x) = 0$. If $\Delta = 11$ or 12, x is adjacent to *eight* $(\geq \Delta-1)$-vertices or *three* Δ-vertices and *five* $(\geq \Delta-2)$-vertices or at least *four* Δ-vertices. So if $\Delta = 11$, x receives at least $\min\{8 \times \frac{1.5}{6}, 3 \times \frac{2.5}{7}, 4 \times \frac{2.5}{7}\} > 0.5$. Hence, $c'(x) \geq 0$. For $\Delta = 12$, x receives at least $\min\{2 \times \frac{2.8}{7} + 6 \times \frac{0.8}{6}, 3 \times \frac{2.8}{7} + 5 \times \frac{0.2}{5}, 4 \times \frac{2.8}{7}\} > 1.2$. Hence $c'(x) \geq 0$.

(IX) Claim that $c'(x) \geq 0$ if $d(x) = 9$.

Note that if $\Delta = 12$, x sends nothing out but receives charges. Since x is adjacent to *nine* (≥ 11)-vertices, or *three* 12-vertices and *six* 10-vertices, or *four* 12-vertices. Hence x receives at least $\min\{9 \times \frac{1.2}{7}, 3 \times \frac{2.8}{8}, 4 \times \frac{2.8}{8}\} > 0.2$, $c'(x) \geq 0$.

Next we consider $\Delta = 10, 11$. By VAL, (R2) and (R3), x sends out at most

$$\begin{cases} [d_{<q}(x)]\dfrac{9-q}{d_{<q}(x)} & \text{if } \delta(x) \neq 7, \text{ or } \Delta \neq 10. \\ 5 \times \dfrac{1}{15} & \text{if } \delta(x) = 7 \text{ and } \Delta = 10. \end{cases}$$

Hence $c'(x) \geq 0$.

(X) Claim that $c'(x) \geq 0$ if $d(x) \geq 10$.

Since $d(x) > q$, by (R2) and (R3), x sends at most $d(x) - q$ out to its neighbors, thus $c'(x) \geq 0$.

From (I)-(X), $c'(x) \geq 0$ for each $x \in V(G)$ and therefore, $\sum_{x\in V(G)} c'(x) \geq 0$. Since the discharge rules only move charge around and do not change the sum, we have $0 \leq \sum_{x\in V(G)} c'(x) = \sum_{x\in V(G)} c(x) < 0$. This contradiction completes the proof.

4. Class one graphs with $c_S = -4, -5, -6$.

The following lemma will be used in the proof of Theorem 4.2

Lemma 4.1 (*Beineke and Fiorini [1], Chetwynd and Yap [3], Brinkmann and Steffen [2]*).

(i) *There are no critical graphs of even order at most 12;*
(ii) *There are only two critical graphs of order 11 with size at most 5Δ, both of which are 3-critical.*

The following theorem is an application of Theorem 3.1.

Theorem 4.2. *Let G be a simple graph that is embeddable in a surface S of characteristic $c_S = -4$, or -5, or -6, then G is class one if $\Delta \geq 10$, or 11, or 12 respectively.*

Proof. By Theorem 3.1 and results listed in the section of Introduction, we only need to prove it when $\Delta = 10, 11, 12$ respectively. Let V and F be vertex set and face set of G respectively. Suppose to the contrary, let G be the smallest counterexample with respect to edges. Then G is Δ-critical where $\Delta = 10, 11, 12$ respectively. By Euler's Formula, we have

$$\begin{cases} \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=24 & \text{if } c_S=-4,\ \Delta=10.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=30 & \text{if } c_S=-5,\ \Delta=11.\\ \sum_{x\in V}(d(x)-6)+\sum_{f\in F}(d(f)-3)=36 & \text{if } c_S=-6,\ \Delta=12.\end{cases}$$

By Theorem 3.1, we have

$$\begin{cases} 2\times|V|\le 24 & \text{if } c_S=-4\Delta=10.\\ 2.5\times|V|\le 30 & \text{if } c_S=-5\Delta=11.\\ 3.2\times|V|\le 36 & \text{if } c_S=-6\Delta=12.\end{cases}$$

Then $|V| \le 11.25$ if $\Delta = 12$, a contradiction. And we have $|V| \le 12$ if $\Delta = 10, 11$. We consider $\Delta = 11$ first. Since $|V| \ge \Delta + 1$, so $|V| = 12$. By Lemma 4.1, there is no 11-critical graph on order 12. Next we consider $\Delta = 10$. Since $|V| \ge \Delta + 1$ and $|V|$ must be odd due to the Lemma 4.1, so $|V| = 11$. By Lemma 4.1 again there is no 10-critical graph on order 11. Hence we have our contradictions and complete our proof. □

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